Calculus of Finite Differences

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Abstract

This paper will introduce Δ notation and use it show basic results concerning finite differences. For $f(x)$, $\Delta f(x)$ is defined as $f(x+1) - f(x)$. Using this, we will show how it relates to differentiation and integration. We will then focus on the properties of polynomials and develop a theory of the Δ transformation as it relates to polynomial functions.

1 Introduction and Defintions

To start, we will give a precise definition of the Δ function. Let $\mathbb{N}_0 = \{n \in \mathbb{Z} | n \geq 0\}$ and let $f : \mathbb{N}_0 \to \mathbb{Z}$ be an arbitrary function. From f, we define a new function $\Delta f : \mathbb{N}_0 \to \mathbb{Z}$ by

$$\Delta f(x) = f(x+1) - f(x).$$

We also define $\Delta^n f$ recursively as

$$\Delta^n f(x) = \Delta(\Delta^{n-1} f(x)).$$

Another notation that will be fundamental to our paper is the binomial coefficient. Rather than viewing it in the traditional sense as

$$\binom{x}{k} = \frac{x!}{k!(x-k)!}$$

for integers $x \geq k$, we will define it as a polynomial in x as

$$\binom{x}{k} = \frac{\prod_{i=0}^{k-1}(x-i)}{k!}$$

where $k! = k(k-1)(k-2)\cdots 2 \cdot 1$ for integers $k \geq 1$ and $k! = 1$ when $k = 0$

2 Numerical Examples

To develop some intuition for the Δ function, we will compute some numerical examples. First consider the set of all polynomials of degree 1. Each of these can be represented in the form $P(x) = ax + b$. The Δ can be computed quite simply as such.

$$\begin{aligned} \Delta P(x) &= [a(x+1) + b] - [ax + b] \\ &= a \end{aligned} \tag{1}$$

Similarly, when $P(x)$ is a degree-2 polynomial, it can be expressed as $P(x) = ax^2 + bx + c$. Therefore,

$$\begin{aligned} \Delta P(x) &= [a(x+1)^2 + b(x+1) + c] - [ax^2 + b + c] \\ &= [ax^2 + x(2a+b) + (a+b+c)] - [ax^2 + bx + c] \\ &= x(2a) + (a+b). \end{aligned} \tag{2}$$

Continuing in this vein, when $P(x) = ax^3 + bx^2 + cx + d$

$$\begin{aligned}\Delta P(x) &= [a(x+1)^3 + b(x+1)^2 + c(x+1) + d] - [ax^3 + bx^2 + cx + d] \\ &= [ax^3 + x^2(3a+b) + x(3a+2b+c) + (a+b+c+d)] - [ax^3 + bx^2 + cx + d] \\ &= x^2(3a) + x^2(3a+2b) + (a+b+c).\end{aligned} \tag{3}$$

Based on these examples, it seems like the Δ of any polynomial of degree n, is a polynomial of degree of $n-1$.

3 Formulas

The Δ function has forumulas analogous to those of differentation.

Addition Formula:

$$\begin{aligned}\Delta(f+g)(x) &= (f+g)(x+1) - (f+g)(x) \\ &= f(x+1) + g(x+1) - f(x) - g(x) \\ &= f(x+1) - f(x) - g(x+1) - g(x) \\ \Delta(f+g)(x) &= \Delta f(x) + \Delta g(x)\end{aligned} \tag{4}$$

Product Formula

$$\begin{aligned}\Delta f(x)g(x) &= f(x+1)g(x+1) - f(x)g(x) \\ &= f(x+1)g(x+1) - f(x+1)g(x) + f(x+1)g(x) - f(x)g(x) \\ &= f(x+1)(g(x+1) - g(x)) + (f(x+1) - f(x))g(x) \\ &= f(x+1)(\Delta g(x)) + (\Delta f(x))g(x) \\ &= g(x+1)(\Delta f(x)) + (\Delta g(x))f(x)\end{aligned} \tag{5}$$

Quotient Formula

$$\begin{aligned}\Delta\left(\frac{f(x)}{g(x)}\right) &= \frac{f(x+1)}{g(x+1)} - \frac{f(x)}{g(x)} \\ &= \frac{f(x+1)g(x) - f(x)g(x+1)}{g(x+1)g(x)} \\ &= \frac{f(x+1)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+1)}{g(x+1)g(x)} \\ &= \frac{(f(x+1) - f(x))g(x) - f(x)(g(x+1) - g(x))}{g(x+1)g(x)} \\ &= \frac{(\Delta f(x))g(x) - f(x)(\Delta g(x))}{g(x+1)g(x)}\end{aligned} \tag{6}$$

Although we weren't able to find a nice formula for Δ of the composition of functions we were able find that for all polynomials $f(x)$ and $g(x)$, $\Delta f(g(x))$ is divisble $\Delta g(x)$. Before we show this we will prove a lemma first.

Lemma: For all polynomials $A(x)$, $B(x)$, and $f(x)$, $A(x) - B(x)$ divides $f(A(x)) - f(B(x))$.

Proof: Since $f(x)$ is a polynomial, it can be expressed in the form

$$f(x) = \sum_{k=0}^{n} c_k x^k \tag{7}$$

Therefore,

$$\begin{aligned} f(A(x)) - f(B(x)) &= \sum_{k=0}^{n} c_k \Big(\big[A(x)\big]^k - \big[B(x)\big]^k \Big) \\ &= \sum_{k=0}^{n} \left[c_k \Big(A(x) - B(x) \Big) \left(\sum_{i=0}^{k-1} A(x)^i B(x)^{k-1-i} \right) \right] \\ &= \Big(A(x) - B(x) \Big) \sum_{k=0}^{n} \left[c_k \left(\sum_{i=0}^{k-1} A(x)^i B(x)^{k-1-i} \right) \right] \end{aligned} \tag{8}$$

which shows that

$$\big(A(x) - B(x)\big) | \big(f(A(x)) - f(B(x))\big)$$

Now, letting

$$\begin{aligned} A(x) &= g(x+1) \\ B(x) &= g(x) \end{aligned} \tag{9}$$

yields

$$g(x+1) - g(x) | f(g(x+1)) - f(g(x))$$

By definition,

$$\begin{aligned} \Delta g(x) &= g(x+1) - g(x) \\ \Delta f(g(x+1)) &= f(g(x+1)) - f(g(x)). \end{aligned} \tag{10}$$

Based on this, we can conclude that

$$\Delta g(x) | \Delta f(g(x))$$

□

3.1 Formula for $\Delta(x^n)$

Let

$$g(x) = x^n.$$

We have defined

$$\Delta f(x) = f(x+1) - f(x).$$

Applying this definition then

$$\Delta g(x) = g(x+1) - g(x)$$
$$\Delta g(x) = (x+1)^n - (x)^n$$

Binomial theorem says that

$$\begin{aligned} (x+y)^n &= \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 \cdots \binom{n}{k} x^{n-k} y^k \cdots \binom{n}{n} x^0 y^n \\ &= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 \cdots \binom{n}{k} x^{n-k} y^k \cdots \binom{n}{n} y^n \\ &= x^n + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 \cdots + \binom{n}{k} x^{n-k} y^k + \cdots + y^n. \end{aligned} \tag{11}$$

If $y = 1$ then the expansion comes out to

$$\begin{aligned} (x+1)^n &= x^n + \binom{n}{1} x^{n-1} 1^1 + \binom{n}{2} x^{n-2} 1^2 \cdots \binom{n}{k} x^{n-k} 1^k \cdots + 1^n \\ &= x^n + \binom{n}{1} x^{n-1} + \binom{n}{2} x^{n-2} \cdots + \binom{n}{k} x^{n-k} + \cdots + 1. \end{aligned} \tag{12}$$

So

$$\begin{aligned}
\Delta g(x) &= x^n + \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}\cdots + \binom{n}{k}x^{n-k} + \cdots + 1 - x^n \\
&= x^n - x^n + \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}\cdots + \binom{n}{k}x^{n-k} + \cdots + 1 \\
&= 0 + \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}\cdots + \binom{n}{k}x^{n-k} + \cdots + 1 \\
&= \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}\cdots + \binom{n}{k}x^{n-k} + \cdots + 1 \\
&= \sum_{i=0}^{n-1}\binom{n}{i}x^i.
\end{aligned} \tag{13}$$

3.2 Formula for $\Delta(f(x))$ where $f(x)$ is any polynomial

Let

$$g(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0$$

where $a_i \in \mathbb{Z}$ and $a_n \neq 0$.

We have defined

$$\Delta(f(x)) = f(x+1) - f(x).$$

We also know that

$$\Delta(f(x) + g(x)) = \Delta(f(x)) + \Delta(g(x))$$

and that

$$\Delta(a_n f(x)) = a_n \Delta(f(x)).$$

So using these two facts we can say

$$\begin{aligned}
\Delta(g(x)) &= \Delta(a_n x^n) + \Delta(a_{n-1}x^{n-1}) + \Delta(a_{n-2}x^{n-2}) + \cdots + \Delta(a_1 x) + \Delta(a_0) \\
&= a_n\Delta(x^n) + a_{n-1}\Delta(x^{n-1}) + a_{n-2}\Delta(x^{n-2}) + \cdots + a_1\Delta(x) + a_0\Delta(1) \\
&= a_n\left[\binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} + \cdots + \binom{n}{k}x^{n-k} + \cdots + 1\right] \\
&\quad + a_{n-1}\left[\binom{n-1}{1}x^{n-1} + \binom{n-1}{2}x^{n-2}\cdots + \binom{n-1}{k}x^{n-k} + \cdots + 1\right] \\
&\quad + a_{n-2}\left[\binom{n-2}{1}x^{n-1} + \binom{n-2}{2}x^{n-2} + \cdots + \binom{n-2}{k}x^{n-k} + \cdots + 1\right] \\
&\quad + \cdots \\
&\quad + a_1.
\end{aligned}$$

Now we group the coefficients to get

$$\begin{aligned}
\Delta(g(x)) = a_n\binom{n}{1}x^{n-1} &+ \left[a_n\binom{n}{2} + a_{n-1}\binom{n-1}{1}\right]x^{n-2} + \cdots + \left[a_n\binom{n}{k} + \cdots + a_{n-1}\binom{n-1}{k-1} + a_{n-k}\binom{n-k}{k-k}\right]x^{n-k} \\
&+ \cdots + \left[a_n + a_{n-1} + a_{n-2} + a_{n-3} + \cdots + a_0\right].
\end{aligned}$$

This can be expressed in summation notation as

$$\begin{aligned}\Delta(g(x)) &= \sum_{k=0}^{n} a_k \Delta(x^n) \\ &= \sum_{k=1}^{n} \left(\sum_{i=1}^{k} a_{n-i+1} \binom{n-i+1}{k-i+1} x^{n-k} \right).\end{aligned}$$

4 Formula for $\Delta f(x)$ When All Coefficients of $f(x)$ Are 1

Let $f(x) = a_{n+1}x^{n+1} + a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$.
As we defined $\Delta(f(X)) = f(x+1) - f(x)$,

$$\begin{aligned}&\Delta(f(x)) \\ &= a_{n+1}[(x+1)^{n+1} - x^{n+1}] + a_n[(x+1)^n - x^n] + ... + a_1[(x+1) - x] \\ &= x^n + x^{n-1} + x^{n-2} + ... + 1 \\ &= a_{n+1}[\binom{n+1}{0} x^{n+1} + \binom{n+1}{1} x^n + ... + \binom{n+1}{n} x + \binom{n+1}{n+1} - x^{n+1}] \\ &+ a_n[\binom{n}{0} x^n + \binom{n}{1} x^{n-1} + ... + \binom{n}{n-1} x + \binom{n}{n} - x^n] \\ &+ a_{n-1}[\binom{n-1}{0} x^{n-1} + \binom{n-1}{1} x^{n-2} + ... + \binom{n-1}{n-2} x + \binom{n-1}{n-1} - x^{n-1}] \\ &+ ... \\ &+ a_2[\binom{2}{0} x^2 + \binom{2}{1} x + \binom{2}{2} - x^2] \\ &+ a_1[\binom{1}{0} x + \binom{1}{1} - x].\end{aligned}$$

Then we simplify a little bit.

$$\begin{aligned}&= a_{n+1}[\binom{n+1}{1} x^n + ... + \binom{n+1}{n} x + \binom{n+1}{n+1}] \\ &+ a_n[\binom{n}{1} x^{n-1} + ... + \binom{n}{n-1} x + \binom{n}{n}] \\ &+ a_{n-1}[\binom{n-1}{1} x^{n-2} + ... + \binom{n-1}{n-1}] \\ &+ ... \\ &+ a_2[\binom{2}{1} x + \binom{2}{2}] \\ &+ a_1[\binom{1}{1}].\end{aligned}$$

Now we group the coefficents according to the powers of x.

$$
\begin{aligned}
&= a_{n+1}\binom{n+1}{1}x^n + [a_{n+1}\binom{n+1}{2} + a_n\binom{n}{1}]x^{n-1} \\
&+ [a_{n+1}\binom{n+1}{3} + a_n\binom{n}{2} + a_{n-1}\binom{n-1}{1}]x^{n-2} \\
&+ [a_{n+1}\binom{n+1}{4} + a_n\binom{n}{3} + a_{n-1}\binom{n-1}{2} + a_{n-2}\binom{n-2}{1}]x^{n-3} \\
&+ ... \\
&+ [a_{n+1}\binom{n+1}{n} + a_n\binom{n}{n-1} + a_{n-1}\binom{n-1}{n-2} + ... + a_2\binom{2}{1}]x \\
&+ [a_{n+1}\binom{n+1}{n+1} + a_n\binom{n}{n} + a_{n-1}\binom{n-1}{n-1} + ... + a_2\binom{2}{2} + a_1].
\end{aligned}
$$

Coefficients of the above are,

$$
\begin{aligned}
&a_{n+1}\binom{n+1}{1} = 1 \\
&a_{n+1}\binom{n+1}{2} + a_n\binom{n}{1} = 1 \\
&a_{n+1}\binom{n+1}{3} + a_n\binom{n}{2} + a_{n-1}1 = 1 \\
&... \\
&a_{n+1}\binom{n+1}{n} + a_n\binom{n}{n-1} + a_{n-1}\binom{n-1}{n-2} + ... + a_2\binom{2}{1} = 1 \\
&a_{n+1}\binom{n+1}{n+1} + a_n\binom{n}{n} + a_{n-1}\binom{n-1}{n-1} + ... + a_2\binom{2}{2} + a_1 1 = 1.
\end{aligned}
$$

From the equations above, first four coeffients are,

$a_{n+1} = 1/(n+1)$
$a_n = (2-n)/(2n)$
$a_{n-1} = (12-(n-1)(6-n))/12(n-1) = (n^2-7n+18)/12(n-1)$
$a_{n-2} = (n^2-9n+26)/12(n-1)$.

5 Integration

A natural question to ask is if there is an antidifference similar to integeration in calculus. Specifically, what is the antidifference of a random polynomial.

5.1 Numerical Examples

Let's start with a basic example. Suppose for all natural numbers n,

$$\Delta f(n) = 3$$

This is equivalent to

$$f(x+1) - f(x) = 3$$

Listing them all out gives

$$\begin{cases} f(1)-f(0)=3 \\ f(2)-f(1)=3 \\ f(3)-f(2)=3 \\ ... \\ f(n)-f(n-1)=3 \end{cases}$$

Adding them yields

$$f(n)-f(0)=3n$$

This method shows a general way in which a function could be contructed on the natural numbers given its Δ values. By definition,

$$\Delta f(n)=f(n+1)-f(n)$$

So

$$\begin{cases} f(1)-f(0)=\Delta f(0) \\ f(2)-f(1)=\Delta f(1) \\ f(3)-f(2)=\Delta f(2) \\ ... \\ f(n)-f(n-1)=\Delta f(n-1) \end{cases}$$

Adding them shows that

$$\begin{aligned} f(x)-f(0)&=\Delta f(0)+\Delta f(1)+\Delta f(2)+...+\Delta f(x-1) \\ f(x)-f(0)&=\sum_{k=0}^{x-1}\Delta f(k). \end{aligned} \tag{14}$$

5.2 Solving $\Delta f(x)=x^n$

Suppose you're given a polynomial $f(x)$ with integer coefficients such that

$$\Delta f(x)=a_n x^n+a_{n-1}x^{n-1}+\cdots+a_0.$$

If a general formula can be found for the antidifference of x^n then the antidifference any polynomial can be constructed. Let k be a nonnegative integer a define the function $\Delta f_n(x):\mathbb{N}_0\to\mathbb{Z}$ by

$$f_n(x)=x^n$$

By our previous result,

$$f(x)-f(0)=0^n+1^n+2^n+...+(x-1)^n \tag{15}$$

We claim that $f(x)$ is a polynomial. Consider the function $g(x)=x^{n+1}$. Based on the result we showed earlier,

$$\begin{aligned} \Delta g(x)&=\binom{n+1}{1}x^n+\binom{n+1}{2}x^{n-1}\cdots+\binom{n+1}{k}x^{n+1-k}+\cdots+1 \\ &=\sum_{i=0}^{n}\binom{n+1}{i}x^i. \end{aligned} \tag{16}$$

Listing all of them out shows

$$\begin{cases} g(1)-g(0)=\binom{n+1}{n}0^n+\binom{n+1}{n-1}0^{n-1}\cdots+\binom{n+1}{k}0^k+\cdots+0^0 \\ g(2)-g(1)=\binom{n+1}{n}1^n+\binom{n+1}{n-1}1^{n-1}\cdots+\binom{n+1}{k}1^k+\cdots+1^0 \\ g(3)-g(2)=\binom{n+1}{n}2^n+\binom{n+1}{n-1}2^{n-1}\cdots+\binom{n+1}{k}2^k+\cdots+2^0 \\ ... \\ g(x)-g(x-1)=\binom{n+1}{n}(x-1)^n+\binom{n+1}{n-1}(x-1)^{n-1}\cdots+\binom{n+1}{k}(x-1)^k+\cdots+(x-1)^0. \end{cases}$$

Adding all of them up and replacing $g(x)$ by x^{k+1} yields

$$\begin{aligned}
x^{k+1} &= \binom{n+1}{n}(0^n + 1^n + 2^n + \cdots + (x-1)^n) + \binom{n+1}{n-1}(0^{n-1} + 1^{n-1} + 2^{n-1} + \cdots (x-1)^{n-1} + \cdots) + \cdots \\
&\quad + \binom{n+1}{0}(0^0 + 1^0 + 2^0 + \cdots + (x-1)^0) \\
&= \sum_{i=0}^{n} \left[\binom{n+1}{i} \sum_{j=0}^{x-1} j^i \right] \\
&= \sum_{i=0}^{n} \left[\binom{n+1}{i} f_i(x) \right] \\
&= \binom{n+1}{n} f_n(x) + \sum_{i=0}^{n-1} \left[\binom{n+1}{i} f_i(x) \right].
\end{aligned} \tag{17}$$

5.3 Solving Using Recursion

The equation above shows a way in which $f_n(x)$ can be calculated recursively by all previous $f_i(x)$. Substracting the summation from both sides and dividing everything by $(n+1)$ shows that

$$\begin{aligned}
f_n(x) &= \frac{x^{k+1} - \sum_{i=0}^{n-1} \left[\binom{n+1}{i} f_i(x) \right]}{n+1} \\
&= \frac{x^{k+1} - \left[\binom{n+1}{n-1} f_{n-1}(x) + \binom{n+1}{n-2} f_{n-2}(x) + \cdots + \binom{n+1}{1} f_1(x) + \binom{n+1}{0} f_0(x) \right]}{n+1}.
\end{aligned} \tag{18}$$

Using the fact that $f_0(x) = n$, we can calculate each of the succesive polynomials.

$$\begin{aligned}
f_0(x) &= x \\
f_1(x) &= \frac{1}{2}x^2 - \frac{1}{2}x \\
f_2(x) &= \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}x \\
f_3(x) &= \frac{1}{4}x^4 - \frac{1}{2}x^3 + \frac{1}{4}x^2 + 0x \\
f_4(x) &= \frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3 + 0x^2 - \frac{1}{30}x \\
f_5(x) &= \frac{1}{6}x^6 - \frac{1}{2}x^5 + \frac{5}{12}x^4 + 0x^3 - \frac{1}{12}x^2 + 0x \\
f_6(x) &= \frac{1}{7}x^7 - \frac{1}{2}x^6 + \frac{1}{2}x^5 + 0x^4 - \frac{1}{6}x^3 + 0x^2 + \frac{1}{42}x \\
f_7(x) &= \frac{1}{8}x^8 - \frac{1}{2}x^7 + \frac{7}{12}x^6 + 0x^5 - \frac{7}{24}x^4 + 0x^3 + \frac{1}{12}x^2 + 0x \\
f_8(x) &= \frac{1}{9}x^9 - \frac{1}{2}x^8 + \frac{2}{3}x^7 + 0x^6 - \frac{7}{15}x^5 + 0x^4 + \frac{2}{9}x^3 + 0x^2 - \frac{1}{30}x.
\end{aligned} \tag{19}$$

These polynomials seem to follow the pattern

$$f_k(x) = \frac{1}{k+1}x^{k+1} - \frac{1}{2}x^k + \frac{k}{12}x^{k-1} + 0x^{k-2} - \frac{\binom{k}{3}}{120}x^{k-3} + 0x^{k-4} + \frac{\binom{k}{5}}{252}x^{k-5} + 0x^{k-6} - \frac{\binom{k}{7}}{240}x^{k-7} + \cdots \tag{20}$$

6 Integration by Parts

Integration by Parts for finite differences is very much like that for Calculus. In Calculus, we all know that, if u and v are real functions that are continuous, then

$$\int u\,dv = uv - \int v\,du.$$

Basic proof is presented below:
By Product Rule:

$$d(uv) = u(dv) + (du)v.$$

Integrate both sides,

$$uv = \int u\,dv + \int du\,v.$$

Rearranging each integral,

$$uv = \int u\,dv + \int v\,du.$$

After moving $\int u\,dv$ to the left of the equation,

$$\int u\,dv = uv - \int v\,du.$$

All the properties that we used to prove the above formula for Calculus are Product Rule, which we have already proved in part 3 just with a different name ”Product Formula”, and the basic properties of any ring such as associativity and communitivity. So Finite Differences satisfy every property that is required for proving integration by parts!
Thus, we can also use a similar proof for Finite Differences!
Let $f(x)$ and $g(x)$ be two functions in Finite Differences. By the Product Formula for Finite Differences,

$$\Delta[f(x)g(x)] = g(x+1)[\Delta f(x)] + [\Delta g(x)]f(x)$$

Integrate both sides,

$$f(x)g(x) = \int g(x+1)\,[\Delta f(x)] + \int [\Delta g(x)]f(x).$$

Put $\int g(x+1)\,[\Delta f(x)]$ to one side,

$$\int g(x+1)\,[\Delta f(x)] = f(x)g(x) - \int [\Delta g(x)]f(x).$$

To make our formula look better,

$$\int g(x+1)\,[\Delta f(x)] = f(x)g(x) - \int f(x)[\Delta g(x)].$$

This is our formula for Integration by Parts.
Let's verify this with a couple of examples. Let $f(x) = x^2 + 2x + 1$ and $g(x) = 3x + 2$.

$$g(x+1) = 3(x+1) + 2$$

$$g(x+1) = 3x + 5$$

$\Delta f(x) = f(x+1) - f(x)$ by the definition of $\Delta f(x)$

$$\Delta f(x) = x^2 + 2x + 1 + 2x + 2 + 1 - x^2 - 2x - 1$$

$$\Delta f(x) = 2x + 3.$$

Multiplying them together,

$$g(x+1)\,[\Delta f(x)] = (3x+5)(2x+3)$$
$$g(x+1)\,[\Delta f(x)] = 6x^2 + 19x + 15.$$

Applying our integration by recursion formula,

$$\int g(x+1)\,[\Delta f(x)] = \int 6x^2 + 19x + 15$$

$$\int g(x+1)\,[\Delta f(x)] = \frac{6x(x+1)(2x+1)}{6} + \frac{19x(x+1)}{2} + 15n + c$$

$$\int g(x+1)\,[\Delta f(x)] = x(x+1)(2x+1) + \frac{19x(x+1)}{2} + 15x + c$$

$$\int g(x+1)\,[\Delta f(x)] = 2x^3 + \frac{25x^2}{2} + \frac{51x}{2}.$$

So now let's evaluate the right side.

$$f(x)g(x) = (x^2 + 2x + 1)(3x + 2)$$

$$f(x)g(x) = 3x^3 + 8x^2 + 7x + 2.$$

Then, we have to evalute $\Delta g(x)$

$$\Delta g(x) = g(x+1) - g(x)$$
$$\Delta g(x) = (3x+5) - (3x+2)$$
$$\Delta g(x) = 3.$$

After that, multiplying $f(x)$ with $\Delta g(x)$,

$$f(x)\Delta g(x) = 3(x^2 + 2x + 1)$$

$$f(x)\Delta g(x) = 3x^2 + 6x + 3$$

Integrating $f(x)\Delta g(x)$,

$$\int f(x)\Delta g(x) = \int 3x^2 + 6x + 3$$

$$\int f(x)\Delta g(x) = \frac{3x(x+1)(2x+1)}{6} + \frac{6x(x+1)}{2} + 3x$$

$$\int f(x)\Delta g(x) = \frac{x(x+1)(2x+1)}{2} + 3x(x+1) + 3x$$

$$f(x)g(x) - \int f(x)[\Delta g(x)] = 3x^3 + 8x^2 + 7x + 2 - \frac{x(x+1)(2x+1)}{2} - 3x(x+1) - 3x$$

$$f(x)g(x) - \int f(x)[\Delta g(x)] = 2x^3 + \frac{25x^2}{2} + \frac{51x}{2}$$

$$f(x)g(x) - \int f(x)[\Delta g(x)] = \int g(x+1)\,[\Delta f(x)].$$

This means that our formula works when $f(x) = x^2 + 2x + 1$ and $g(x) = 3x + 2$.

7 Formula for $\Delta^n(f(x)) = c$

So far, we have only examined the properties of a single antidifference and recursively found the antidifference for polynomials of the form x^k for which k is a natural number. But how how would a polynomial be transformed if the antidifference is applied more than once? In other words, given a polynomial $P(x)$, for a natural number n what would be the solution to

$$\begin{aligned}\Delta^n f(x) &= P(x) \\ &= \sum_{i=0}^{k} a_i x^k.\end{aligned} \tag{21}$$

To tackle this problem, a first step would be to find the general solution to

$$\Delta^n f(x) = c. \tag{22}$$

Based on our previous results, letting

$$n = 1 \Rightarrow f(x) = cx + c_0$$

Now let us consider the case of when $n = 2$.

$$\Delta^2 f(x) = c. \tag{23}$$

This, by definition, is equivalent to

$$\Delta(\Delta f(x)) = c. \tag{24}$$

Combining this with the result for $n = 1$ implies

$$\Delta f(x) = cx + c_0. \tag{25}$$

Listing all of the terms out gives

$$\begin{cases} f(1) - f(0) = c\binom{0}{1} + c_0 \\ f(2) - f(1) = c\binom{1}{1} + c_0 \\ f(3) - f(2) = c\binom{2}{1} + c_0 \\ \cdots \\ f(n) - f(n-1) = c\binom{n-1}{1} + c_0 \end{cases}$$

Adding them all up and using the hockey-stick identity shows that a solution to $\Delta^2 f(x) = c$ is

$$\begin{aligned} f(x) &= \left[c\binom{n-1}{1} + c_0\right] + \left[c\binom{n-2}{1} + c_0\right] + \cdots + \left[c\binom{2}{1} + c_0\right] + \left[c\binom{1}{1} + c_0\right] + \left[c\binom{0}{1} + c_0\right] + f(0) \\ &= c\binom{n}{2} + c_0\binom{n}{1} + f(0). \end{aligned} \tag{26}$$

These first 2 examples seem to show that for all natural numbers n, the solution to $\Delta^n f(x) = c$ is

$$f(x) = c\binom{x}{n} + c_{n-1}\binom{x}{n-1} + c_{n-2}\binom{x}{n-2} + \cdots + c_1\binom{x}{1} + c_0. \tag{27}$$

Proof: We will show the result by induction. The base case for n = 1 has already been established. Suppose that the solution to $\Delta^k f(x) = x$ is

$$f(x) = c\binom{x}{k} + c_{k-1}\binom{x}{k-1} + c_{k-2}\binom{x}{k-2} + \cdots + c_1\binom{x}{1} + c_0. \tag{28}$$

This would imply that the solution to $\Delta^{k+1}f(x) = x$ is equivalent to the solution of

$$\Delta f(x) = c\binom{x}{k} + c_{k-1}\binom{x}{k-1} + c_{k-2}\binom{x}{k-2} + \cdots + c_1\binom{x}{1} + c_0. \tag{29}$$

Like in the case of $n = 2$, list out all of the terms and add them up using the hockeystick identity to obtain the solution

$$\Delta f(x) = c\binom{x}{k+1} + c_k\binom{x}{k} + c_{k-1}\binom{x}{k-1} + \cdots + c_1\binom{x}{2} + c_0\binom{x}{1} + f(0). \tag{30}$$

This equation is of the form we asserted it would be, so we are done.
Unfortunately, we're not sure how this solution would generalize.

8 Line Fitting

One problem that can be solved using Δ is contructing a polynomial $P(x)$ with the points $(0, y_0), (1, y_1), (2, y_2), \cdots, (n-1, y_{n-1}), (n, y_n)$. Based on this, we can calculate the Δ values of $P(x)$.

$$\Delta P(0) = P(1) - P(0), \Delta P(1) = P(2) - P(1), \cdots, \Delta P(n-1) = P(n) - P(n-1)$$

$$\Delta^2 P(0) = \Delta P(1) - \Delta P(0), \Delta^2 P(1) = \Delta P(2) - \Delta P(2), \cdots, \Delta^2 P(n-2) = \Delta P(n-1) - \Delta P(n-2)$$

$$\cdots$$

$$\Delta^n P(0) = \Delta^{n-1} P(1) - \Delta^{n-1} P(0)$$

A polynomial $P(x)$ that has all of the points is

$$P(x) = \Delta^n P(0)\binom{x}{n} + \Delta^{n-1}P(0)\binom{x}{n-1} + \cdots + \Delta^2 P(0)\binom{x}{2} + \Delta P(0)\binom{x}{1} + P(0)\binom{x}{0}. \tag{31}$$

There is another way in which a polynomial can be generated. The points are satisfied by the polynomial

$$\Delta f(x) = y_0 q_0(x) + y_1 q_1(x) + \cdots + y_{n-1} q_{n-1}(x) + y_n q_n(n) \tag{32}$$

where each $q_i(x)$ is of the form

$$q_i(x) \;=\; \prod_{k=0, k\neq i}^{n} \frac{x-i}{i-k}.$$

This works because $q_i(x) = 0$ for all integers $x \in [0, n]$ except for i. $x = i \Rightarrow q_i(x) = 1$. So $P(i) = y_i$.

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