Building a Tower

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Can we determine the optimal shape of a tower in terms of wind resistance? In this paper, we will derive the equations defining the optimal shape in two dimensional space, and three dimensional space, based on the Eiffel Tower property, stated by Gustave Eiffel.

INTRODUCTION

Consider a tower defined by two symmetric curves as shown in FIG. 1. The Eiffel Tower property states that the tower is stable if the tangent line at any height h to both sides of the tower intersects at the center of mass of the portion of the tower above h. Using this property, we will be deriving equations determining the equations of the curves.

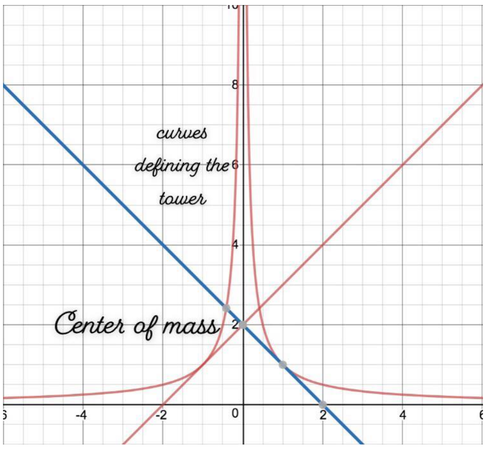


FIG. 1: Curves defining the shape of the tower and two tangent lines taken at $h = 1$

DETERMINING THE SHAPE IN 2D

Suppose some $f(x)$ and $-f(x)$ define the shape of our tower. Assuming the mass is distributed evenly with area, we can easily calculate the location of the center of mass $(0, \bar{y})$.

$$(0, \bar{y}) = \frac{\text{moment}}{\text{area}} = \frac{2\int_h^b yf(y)dy}{2\int_h^b f(y)dy} \tag{1}$$

As one of the restrictions on the graph has to do with the tangent line at a certain height, h, the slope at this height is also a factor that restricts $f(x)$, and we can use $(0, \bar{y})$ and $(f(h), h)$ to construct this slope.

$$f'(h) = \frac{-f(h)}{\bar{y} - h} \tag{2}$$

This equation can be rewritten as

$$f'(h)(\bar{y} - h) = -f(h) \tag{3}$$

and now plugging in Equation (1), we have

$$f'(h)(\frac{2\int_h^b yf(y)dy}{2\int_h^b f(y)dy} - h) = -f(h) \tag{4}$$

In order to make this differential equation more convenient to solve, we can write $f(y)$ in terms of $A(h)$, the area of the tower above height h.

$$A(h) = 2\int_h^b f(y)dy \tag{5}$$

By the Fundamental Theorem of Calculus,

$$A'(h) = 2\frac{d}{dh}(\int_h^b f(y)dy) \tag{6}$$

$$A'(h) = -2f(h) \tag{7}$$

$$f(h) = -\frac{1}{2}A'(h) \tag{8}$$

$$f'(h) = -\frac{1}{2}A''(h) \tag{9}$$

Now plugging Equation (8), (9), and (5) into Equation (4), we get

$$A''(h)(\frac{2\int_h^b yf(y)dy}{A(h)} - h) = A'(h) \tag{10}$$

By simple algebra, we can isolate this integral to one side, and obtain

$$2\int_h^b yf(y)dy = (-\frac{A'(h)}{A''(h)} + h)A(h) \tag{11}$$

Taking derivative once again,

$$-2hf(h) = A'h + A - \frac{A''(A''A + A'A') - A'AA'''}{(A'')^2} \tag{12}$$

which simplifies to

$$A''A' - AA''' = 0 \tag{13}$$

We can rewrite this as

$$\frac{A'''}{A''} = \frac{A'}{A} \tag{14}$$

and integrate.

$$\int \frac{A'''}{A''} dh = \int \frac{A'}{A} dh \tag{15}$$

$$ln(A'') = ln(A) + C_1 \tag{16}$$

$$A'' - C_2 A = 0 \tag{17}$$

where C_1 and C_2 are some constants.
$A = Ce^{rt}$,$A = Csin(bt)$, and $A = Ccos(bt)$ where c,r, and b are mathematically acceptable solutions, and now that we have A, we can solve for $f(y)$.

$$f(y) = -\frac{1}{2} cre^{ry} \tag{18}$$

Note while $A = Ce^{rt}$,$A = Csin(bt)$, and $A = Ccos(bt)$ where c,r, and b are all mathematically acceptable, only the most realistic, and feasible, physical solution $f(y) = -\frac{1}{2}cre^{ry}$ when $A = Ce^{rt}$ is shown here.

DETERMINING THE SHAPE IN 3D

As towers in real life are not two dimensional, we will now look at three dimensional towers by revolving $y = f(x)$ by y axis. Similarly, the coordinate of center of mass will be $(0, \bar{y}, 0)$. As we can look at the tower as a pile of discs with radius $f(y)$ and height dy, moment is

$$\text{moment} = \int\int\int_V y dV = \int_h^b y\pi f^2(y) dy \tag{19}$$

giving

$$\bar{y} = \frac{\pi \int_h^b y f^2(y) dy}{V(h)} \tag{20}$$

As

$$V(h) = \int_h^b \pi f^2(y) dy \tag{21}$$

once again, by Fundamental Theorem of Calculus, we get

$$V'(h) = -\pi f^2(h) \tag{22}$$

and we can rewrite this

$$f(h) = -\sqrt{\frac{V'(h)}{\pi}} \tag{23}$$

as $f(h) > 0$. Taking derivative, we obtain

$$f'(h) = \frac{-\frac{V''(h)}{\pi}}{2\sqrt{\frac{-V'(h)}{\pi}}} = \frac{-V''(h)}{2\pi\frac{-V'(h)}{\pi}} \tag{24}$$

The slope at h is $f'(h) = \frac{-f(h)}{y-h}$. Note that this is the same as the case of plane because for any point in the circle, which was created by revolving the point $(f(h), h)$ around the y axis, its tangent meets $(0, \bar{y})$. $(f(h), h)$ is a special case. As $f'(h) = \frac{-f(h)}{y-h}$,

$$f'(h)(\bar{y} - h) = -f(h) \tag{25}$$

Now plugging Equation (20), (23), and (24) into Equation (25), we get

$$\frac{-V''(h)}{2\pi\frac{-V'(h)}{\pi}}\left(\frac{\pi\int_h^b y f^2(y) dy}{V(h)} - h\right) = -\sqrt{\frac{-V'(h)}{\pi}} \tag{26}$$

$$(\pi \int_h^b y f^2(y) dy - hV)V'' = -2\pi \frac{V'}{-\pi} V \tag{27}$$

$$(\pi \int_h^b y f^2(y) dy - hV)V'' = -2V'V \tag{28}$$

$$\pi \int_h^b y f^2(y) dy = hV - \frac{2V'V}{V''} \tag{29}$$

Taking derivative, we have

$$-\pi h f^2(h) = V + hV' - \frac{2(V''V + V'^2V'' - V'VV''')}{V''^2} \tag{30}$$

Note the left hand side of Equation (30) is hV', so cancelling out hV' on both sides, we get,

$$VV''^2 - 2(V''^2V + V'^2V'' - V'VV''') = 0 \tag{31}$$

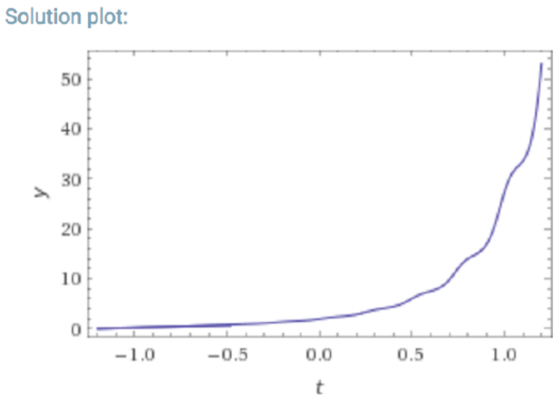
$$2V'VV''' = VV''^2 + 2V'^2V'' \tag{32}$$

$$2\frac{V}{V'}\frac{V'''}{V''} = \frac{V}{V'}\frac{V''}{V'} + 2 \tag{33}$$

$$2\frac{V'''}{V''} = \frac{V''}{V'} + 2\frac{V'}{V} \tag{34}$$

$$\int 2\frac{V'''}{V''} dh = \int \frac{V''}{V'} dh + \int 2\frac{V'}{V} dh \tag{35}$$

$$2ln(V'') = ln(V') + 2ln(V) + C \tag{36}$$



$$V''^2 = V'V^2e^C \quad (37)$$

Let $e^C = k$ then we have

$$V''^2 = kV'V^2 \quad (38)$$

where $k > 0$ which is a nonlinear differential equation.

We cannot solve this nonlinear differential equation as we have previously solved linear differential equations but we can use Euler's method to approximate.

We can also guess for constant k as whenever wind hits the building hard enough, the building slightly tilts, and there will be a restoring torque. In real life, this would take many factors, i.e.force of wind at different heights, and radius of the base of the tower, into account but we will not discuss the physical aspects of this model in this paper.

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